Mean-square convergence of Fourier series

ACM 07

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The inner product

Define an operation (inner product) on the class of complex-valued 2π-periodic and Riemann integrable functions

$$(f, g) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\overline{g(\theta)} d\theta.$$

Particularly,

$$(f, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(\theta)|^2 d\theta = \|f\|_{L^2(\mathbb{T})}^2.$$

The representation of Fourier series

Introduce the orthogonal system $\{e_m\}_{m\in\mathbb{Z}}$

$$e_m(\theta) = e^{im\theta}.$$

The N**-th partial sum of the Fourier series of** f

$$\begin{aligned} S_N(f)(\theta) &= \sum_{|m|\le N} \widehat{f}(m) e^{im\theta} \\ &= \sum_{|m|\le N} \frac{1}{2\pi}\int_0^{2\pi} f(y)e^{-imy}dy \cdot e^{im\theta} \\ &= \sum_{|m|\le N} (f, e_m)e^{im\theta}. \end{aligned}$$

The orthogonality

Basic lemma

$$(f - S_N(f)) \perp e_m \quad (|m| \leq N)$$

Corollary 1: (The Pythagorean theorem)

$$(f - S_N(f)) \perp S_N(f)$$
$$\|f\|^2_{L^2(\mathbb{T})} = \|f - S_N(f)\|^2_{L^2(\mathbb{T})} + \|S_N(f)\|^2_{L^2(\mathbb{T})}$$

Corollary 2: (Best approximation)

$$\|f - S_N(f)\|_{L^2(\mathbb{T})} \leq \|f - P\|_{L^2(\mathbb{T})} \quad \Big((Degree)(P) \leq N\Big)$$

The mean-square convergence of Fourier series

Case 1: Continuous functions
(Tools: The Weierstrass trigonometric polynomial theorem & Best approximation)

Case 2: Riemann integrable functions

The Parseval identity and the Riemann-Lebesgue lemma

The Parseval identity follows from the square-mean convergence and Corollary 1:

$$\|f\|^2_{L^2(\mathbb{T})} = \sum_{m\in\mathbb{Z}} |\widehat{f}(m)|^2.$$

The Riemann-Lebesgue lemma follows from the Parseval identity:

$$\widehat{f}(m) \to 0 \quad (|m| \to \infty)$$

Wonderful applications

A wonderful application:

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

You can discover many new formulas!

Have a rest for a moment!

Another inner product on the other function space

Define an operation (inner product) on the class of **real-valued** **Riemann integrable functions on** $[0, \pi]$

$$(f, g) = \frac{1}{\pi} \int_0^{\pi} f(x)g(x)dx$$

Particularly,

$$(f, f) = \frac{1}{\pi} \int_0^{\pi} |f(x)|^2 dx.$$

The orthogonal system

Introduce the orthogonal system $\{e_n\}_{n\in\mathbb{N}}$

$$e_n(x) = \sqrt{2}\,\mathsf{sin}(nx).$$

Define the N**-th partial sum of the "Fourier series" of** f

$$S_N(f) = \sum_{n=1}^{N} (f, e_n)e_n.$$

The Bessel inequality

Combining

$$\big(S_N(f), S_N(f)\big) = \sum_{n=1}^{N} \big|(f, e_n)\big|^2$$

with the Pythagorean theorem

$$(f, f) = \big(S_N(f), S_N(f)\big) + \big(f - S_N(f), f - S_N(f)\big)$$

yields the Bessel inequality

$$\sum_{n=1}^{\infty} \big|(f, e_n)\big|^2 \leq (f, f).$$

An open question for 07 ACMer

Can you prove or disprove

$$\sum_{n=1}^{\infty} \left|(f, e_n)\right|^2 = (f, f)?$$

Do you have such a puzzel: Where is the cosine?!
I believe the resolution of this question could help you understanding more better the structure of the inner product spaces and their orthogonal systems.

From Fourier to Haar

The Haar system

Define an operation (inner product) on the class of real-valued Riemann integrable functions on $[0,1]$

$$(f,g)=\int_0^1 f(x)g(x)dx.$$

Particularly,

$$(f,f)=\int_0^1 |f(x)|^2dx.$$

The Haar system

The basic Haar function

$$\Psi(x) = \textbf{sign}(\frac{1}{2} - x) \quad (0 \le x \le 1)$$

The Haar system

$$\Psi(x),$$
$$\sqrt{2}\Psi(\frac{x}{2}), \sqrt{2}\Psi(\frac{x}{2} - \frac{1}{2}),$$
$$2\Psi(\frac{x}{4}), 2\Psi(\frac{x}{4} - \frac{1}{4}), 2\Psi(\frac{x}{4} - \frac{2}{4}), 2\Psi(\frac{x}{4} - \frac{3}{4}),$$
$$\cdots, \cdots, \cdots, \cdots, \cdots, \cdots, \cdots, \cdots, \cdots, \cdots, \cdots, \cdots$$

The Haar system

Define the N-th partial sum of the Haar series of f

$$S_N(f) = \sum_{n=1}^{2^N-1} (f, e_n) e_n.$$

The Haar system

Theorem 1

$$\lim_{N\to\infty}\int_0^1 |f - S_N(f)|^2 dx = 0$$

holds for any Riemann integrable function f.

Theorem 2

$$\lim_{N\to\infty}\left(\sup_{0\le x\le 1}\left|f - S_N(f)\right|\right) = 0$$

holds for any continuous function g.

The advantages (square integrable functions and continuous functions) and defects (smooth functions) of Haar series

This would open the window of wavelet analysis.